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A Dual Representation for Proper Positively Homogeneous Functions

MARCO CASTELLANI
Dipartimento di Sistemi ed Istituzioni per l'Economia, Università degli Studi di L'Aquila, Via Assergi 6, 67100 L'Aquila, Italy (e-mail: castella@ec.univaq.it)

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Abstract. In this paper we show that any proper positively homogeneous function annihilating at the origin is a pointwise minimum of sublinear functions (MSL function). By means of a generalized Gordan's theorem for inequality systems with MSL functions, we present an application to a locally Lipschitz extremum problem without constraint qualifications.

Key words: MSL functions; Generalized Gordan's theorem; Necessary optimality condition

1. Introduction and dual characterization

Glover et al. presented in [3] a generalized Farkas lemma for inequality systems with pointwise minimum of sublinear functions (MSL functions). They also stated that such a class of functions is very wide. The aim of this paper is to show that the class of the MSL functions is equivalent to the class of the proper positively homogeneous functions which are zero at the origin. This, together with a generalized Gordan's theorem for MSL functions, paves the way to deriving a necessary optimality condition in dual form for a locally Lipschitz extremum problem without any constraint qualification.

We briefly give the notation used below. $\mathbb{X}$ is a real normed vector space with the norm $\|\cdot\|$, $\mathbb{X}^*$ is its topological dual space endowed with the weak* topology and $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between $\mathbb{X}^*$ and $\mathbb{X}$. We denote by $\mathbb{S}$ the unit sphere; for a set A, we denote the closure, the interior, the convex hull, and the conical hull of A by $\mathrm{cl}\, A$, $\mathrm{int}\, A$, $\mathrm{conv}\, A$, and $\mathrm{cone}\, A$, respectively. The recession cone of A is

$$0^+A := \{x \in \mathbb{X} : A + x \subseteq A\} .$$

The indicator function associated to A is defined by

$$\delta(x, A) := \begin{cases} 0 , & \text{if } x \in A , \\ +\infty , & \text{if } x \not\in A \end{cases}$$

while, if A is a subset of $\mathbb{X}^*$, the support function associated to A is

$$\sigma(x, A) := \sup_{x^* \in A} \langle x^*, x \rangle .$$

The polar cone associated to a cone K in $\mathbb{X}$ is

$$K^\circ := \{x^* \in \mathbb{X}^* : \langle x^*, x\rangle \leq 0, \text{ for all } x \in K\}.$$

A function $p : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ is proper if dom $p \neq \emptyset$, where

$$\text{dom } p := \{x \in \mathbb{X} : p(x) < +\infty\}.$$

If the function p is the support function of a closed convex set A, then its domain is called barrier cone of A and it is denoted by barr $A :=$ dom $\sigma(\cdot, A)$. The function p is sublinear if it is proper convex and positively homogeneous. If p is lower semicontinuous then [7] there exists a unique nonempty closed convex subset of $\mathbb{X}^*$, called subdifferential and denoted by $\partial p(0)$, such that $p(x) = \sigma(x, \partial p(0))$. Lastly, let $\varepsilon > 0$ be fixed; the ε-subdifferential of p at $x_0 \in$ dom p is the closed convex set

$$\partial_\varepsilon p(x_0) := \{x^* \in \mathbb{X}^* : p(x) \geq p(x_0) + \langle x^*, x - x_0\rangle - \varepsilon, \text{ for all } x \in \mathbb{X}\}.$$

The ε-subdifferential of lower semicontinuous sublinear functions at zero coincides with the subdifferential.

DEFINITION 1. *A function $p : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ is said to be pointwise minimum of sublinear functions (MSL function) if there exist an index set T and a family $\{p_t : t \in T\}$ of lower semicontinuous proper sublinear functions such that*

$$p(x) = \min_{t \in T} p_t(x), \quad \textit{for all } x \in \mathbb{X}.$$

By means of the result in [7] the definition of MSL function is equivalent to the existence of a family $\{\partial(p, t) : t \in T\}$ of nonempty closed convex sets in $\mathbb{X}^*$ such that

$$p(x) = \min_{t \in T} \sigma(x, \partial(p, t)), \quad \text{for all } x \in \mathbb{X}.$$

Clearly an MSL function is positively homogeneous and it is zero at the origin. The concept of MSL function falls under the more general definition of *inf-convexity* that was used by Kutateladze and Rubinov [8]. In [1] Crouzeix showed that, if p is a continuous and quasiconvex positively homogeneous function on $\mathbb{R}^n$, then p is an MSL function with the index set T including only two elements. The converse does not hold; indeed the function $p(x) := -|x|$ defined on $\mathbb{R}$ is an MSL function with respect to $T = \{\pm 1\}$ and $p_{\pm 1}(x) = \pm x$ but it is not quasiconvex. Lastly, in [3] Glover et al. showed that the class of MSL functions includes the class of difference sublinear functions. Now we show that the class of MSL functions is very wide: it is equivalent to the set of the positively homogeneous functions annihilating at the origin.

THEOREM 1. *Let $p : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ be a positively homogeneous function such that $p(0) = 0$; then p is an MSL function.*

Proof. If dom $p = \{0\}$, then we choose T a singleton and $\partial p(0) := \mathbb{X}^*$; otherwise let $T := \mathbb{S} \cap$ dom $p \neq \emptyset$ and, for each $z \in T$, we define

$$\partial(p, z) = (\text{cone}\{z\})^{\circ} + p(z)z^*$$

where $z^* := z^*(z) \in \mathbb{X}^*$ is chosen so that $\langle z^*, z\rangle = 1$. Then, for each $x \in \mathbb{X}$,

$$\sigma(x, \partial(p, z)) = p(z)\langle z^*, x\rangle + \sigma(x, (\text{cone}\{z\})^{\circ}) = p(z)\langle z^*, x\rangle + \delta(x, \text{cone}\{z\}) ,$$

where the last equality descends from the relation $\sigma(\cdot\,, K^{\circ}) = \delta(\cdot\,, K)$ which holds for each closed convex cone K. Thus

$$\min_{z\in T} \sigma(x, \partial(p, z)) = \min_{z\in T} \{p(z)\langle z^*, x\rangle + \delta(x, \text{cone}\{z\})\} = p(x) ,$$

and the theorem is proved. □

REMARK 1. Theorem 1 holds also for $\mathbb{X}$ a separated locally convex space. Fixed $z \in \text{dom}\, p$, consider the function

$$p_z(x) : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$$

defined by

$$p_z(x) = \begin{cases} tp(z) , & \text{if } x = tz \text{ with } t \geqslant 0 , \\ +\infty , & \text{otherwise} . \end{cases}$$

Then, the function p_z is lower semicontinuous and sublinear; moreover

$$p(x) = \min_{z\in \text{dom}\, p} p_z(x) .$$

The representation that we have just given might seem too general, in the sense that the index set T might appear too big. Actually, the following example shows that there exist MSL functions such that each index set T has cardinality not less than that of $\mathbb{S} \cap \text{dom}\, p$.

EXAMPLE 1. Let $\mathbb{X} := \mathbb{R}^2$ and, for each nonzero $x \in \mathbb{R}^2$, set α_x the unique element of $[0, 2[$ such that $x = \|x\|(\cos(\alpha_x \pi), \sin(\alpha_x \pi))$. Consider the following positively homogeneous function $p : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$p(x) := \begin{cases} 0 , & \text{if } x = (0, 0) \text{ or } \alpha_x \not\in \mathbb{Q} , \\ \|x\| , & \text{if } \alpha_x \in \mathbb{Q} . \end{cases}$$

Clearly, from Theorem 1, p is MSL and we show that we can not associate the same subdifferential $\partial(p, t)$ to different points $x_1, x_2 \in \mathbb{S}$ with $0 < \alpha_{x_1} < \alpha_{x_2} < 1$ and $\alpha_{x_1}, \alpha_{x_2} \not\in \mathbb{Q}$. If, by contradiction, there exists $\partial(p, t)$ such that

$$\sigma(x_1, \partial(p, t)) = \sigma(x_2, \partial(p, t)) = 0 ,$$

we have

$$\sigma(\lambda x_1 + (1-\lambda)x_2, \partial(p, t)) = 0 , \quad \text{for all } \lambda \in [0, 1] .$$

Since $x_\lambda := \lambda x_1 + (1-\lambda)x_2 \neq (0, 0)$ and $f(x_\lambda) = 0$, then $\alpha_{x_\lambda} \not\in \mathbb{Q}$ for all $\alpha_{x_\lambda} \in$

$[\alpha_{x_1}, a_{x_2}]$ which contradicts the density of $\mathbb{Q}$. Therefore the cardinality of every index set T is not less than that of $\mathbb{S}$.

We conclude proving that for a Lipschitz positively homogeneous function p, the functions p_t may be chosen to be continuous or, equivalently, the sets $\partial(p,t)$ may be chosen compact.

THEOREM 2. *Let $p:\mathbb{X}\to\mathbb{R}$ be a Lipschitz positively homogeneous function with constant k. Then there exist an index set T and a family $\{p_t : t\in T\}$ of continuous proper sublinear functions such that*

$$p(x)=\min_{t\in T} p_t(x)\,, \quad \textit{for all } x\in\mathbb{X}\,.$$

Proof. Since dom $p=\mathbb{X}$, for every $z\in\mathbb{S}$ the function p_z defined in Remark 1 is a lower semicontinuous proper sublinear function. Consider the function

$$\begin{aligned} p^z(x) &= \inf\{k\|x_1\| + p_z(x_2) : x_1 + x_2 = x\} \\ &= \inf\{k\|x-\bar{x}\| + p_z(\bar{x}) : \bar{x}\in\mathbb{X}\} \\ &= \inf\{k\|x-tz\| + tp(z) : t\geqslant 0\}\,. \end{aligned}$$

Since p^z is the infimal convolution of two proper sublinear functions with one of them continuous, then it is a continuous sublinear function. From the Lipschitz assumption, we have

$$p(x)\leqslant k\|x-tz\| + p(tz)\,, \quad \text{for all } t\geqslant 0 \text{ and } x\in\mathbb{X}$$

and therefore

$$p^z(x)\geqslant p(x)\,, \quad \text{for all } x\in\mathbb{X}\,.$$

Moreover, if $x\in\mathbb{S}$, choosing $z=x$ and $t=1$, we have

$$p^x(x) = \inf\{k\|x-tx\| + tp(x) : t\geqslant 0\}\leqslant p(x)\,,$$

which implies

$$p(x) = p^x(x) = \min_{z\in\mathbb{S}} p^z(x)\,, \quad \text{for all } x\in\mathbb{S}\,.$$

Therefore, for each $x\in\mathbb{X}\backslash\{0\}$,

$$p(x) = \|x\|p(\|x\|^{-1}x) = \|x\|p^{\|x\|^{-1}x}(\|x\|^{-1}x) = \|x\|\min_{z\in\mathbb{S}} p^z(\|x\|^{-1}x) = \min_{z\in\mathbb{S}} p^z(x)$$

which proves the statement. □

2. Application to nonsmooth programming problem

Recently some solvability results involving inf-convex functions were presented in [4] and, more specifically, a generalization of the Farkas lemma for MSL systems was proved in [3]. The aim of this section is to state a generalization of the Gordan's theorem [5] and to apply it to a locally Lipschitz extremum problem. In what follows we assume that $I := \{1, \dots, m\}$ is a finite index set, and $p_i : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$, $i \in I_0 := \{0\} \cup I$, are MSL functions such that there exist families $\{\partial(p_i, t_i) : t_i \in T_i\}$, $i \in I_0$, of nonempty closed convex subsets of $\mathbb{X}^*$ with

$$p_i(x) := \min_{t_i \in T_i} \sigma(x, \partial(p_i, t_i)) , \quad \text{for all } i \in I_0 . \tag{1}$$

THEOREM 3. *Assume that for each $i \in I_0$, $p_i : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ is an MSL function with (1) being satisfied; then the following statements are equivalent:*

(i) it is impossible the system

$$p_i(x) < 0 , \quad i \in I_0 ;$$

(ii) for each $(t_0, t_1, \dots, t_m) \in \Pi_{i \in I_0} T_i$,

$$0 \in \operatorname{cl\,conv} \bigcup_{i \in I_0} \partial(p_i, t_i) .$$

Moreover, if barr $\partial(p_i, t_i)$ *are closed statement (i) is equivalent to the following one:*

(iii) for each $(t_0, t_1, \dots, t_m) \in \Pi_{i \in I_0} T_i$, there exist $\lambda_i \geqslant 0$, $i \in I_0$, not all zero, such that

$$0 \in \operatorname{cl}\Big(\sum_{i \in I_0, \lambda_i > 0} \lambda_i \partial(p_i, t_i) + \sum_{i \in I_0, \lambda_i = 0} 0^+ \partial(p_i, t_i) \Big) .$$

Proof. Statement (i) is equivalent to the impossibility of the system

$$\sigma(x, \partial(p_i, t_i)) < 0 , \quad i \in I_0 \tag{2}$$

for each fixed $t := (t_0, t_1, \dots, t_m) \in \Pi_{i \in I_0} T_i$. Define

$$P_t(x) := \max_{i \in I_0} \sigma(x, \partial(p_i, t_i)) = \sigma(x, \operatorname{cl\,conv} \bigcup_{i \in I_0} \partial(p_i, t_i)) ;$$

therefore the impossibility of (2) is equivalent to affirm that P_t assumes global minimum at the origin and thus

$$0 \in \partial P_t(0) = \operatorname{cl\,conv} \bigcup_{i \in I_0} \partial(p_i, t_i)$$

which is statement (ii). Let us consider (i) and (iii). The impossibility of (2) is also equivalent to the disjunction between the interior of the negative orthant $\mathbb{R}^{1+m}_-$ and the convex cone

$$\mathscr{C} = \{y \in \mathbb{R}^{1+m} : \text{there exists } x \in \mathbb{X} \text{ s.t. } \sigma(x, \partial(p_i, t_i)) \leqslant y_i, \text{ for all } i \in I_0\} .$$

By the separation theorem, there exists a nonzero vector $\lambda \in \mathbb{R}^{1+m}$ such that

$$\sum_{i \in I_0} \lambda_i z_i \leqslant 0 \leqslant \sum_{i \in I_0} \lambda_i \sigma(x, \partial(p_i, t_i)) ,$$

$$\text{for all } z \in \operatorname{int} \mathbb{R}^{1+m}_- \text{ and } x \in \bigcap_{i \in I_0} \operatorname{barr} \partial(p_i, t_i) .$$

From the first inequality we deduce $\lambda_i \geqslant 0$, from the second inequality we obtain that the function

$$F(x) := \sum_{i\in I_0, \lambda_i>0} \lambda_i \sigma(x, \partial(p_i, t_i)) + \sum_{i\in I_0, \lambda_i=0} \delta(x, \text{barr}\, \partial(p_i, t_i))$$

assumes global minimum at the origin and then

$$0 \in \partial\Big(\sum_{i\in I_0, \lambda_i>0} \lambda_i \sigma(\cdot\,, \partial(p_i, t_i)) + \sum_{i\in I_0, \lambda_i=0} \delta(\cdot\,, \text{barr}\, \partial(p_i, t_i))\Big)(0)\,. \tag{3}$$

The closure of the sets barr $\partial(p_i, t_i)$ implies that F is sum of lower semicontinuous functions and thus we may apply to (3) the formula derived in [6, Theorem 2.1], deriving

$$0 \in \bigcap_{\varepsilon>0} \text{cl}\Big(\sum_{i\in I_0, \lambda_i>0} \partial_\varepsilon \sigma(0, \lambda_i \partial(p_i, t_i)) \sum_{i\in I_0, \lambda_i=0} \partial_\varepsilon \delta(0, \text{barr}\, \partial(p_i, t_i))\Big)\,;$$

therefore

$$0 \in \bigcap_{\varepsilon>0} \text{cl}\Big(\sum_{i\in I_0, \lambda_i>0} \lambda_i \partial(p_i, t_i)) \sum_{i\in I_0, \lambda_i=0} (\text{barr}\, \partial(p_i, t_i))^\circ\Big)\,.$$

Since ε does not appear into the intersection, and $(\text{barr}\, \partial(p_i, t_i))^\circ = 0^+ \partial(p_i, t_i)$, we achieve the thesis. □

Let us apply Theorem 3 to the inequality constrained extremum problem

$$\min\{f_0(x) : f_i(x) \leqslant 0,\ i \in I := \{1, \ldots, m\}\} \tag{4}$$

where $f_i : \mathbb{X} \to \mathbb{R}$ are locally Lipschitz functions, $i \in I_0 := \{0\} \cup I$. We recall that the *upper* and *lower Dini directional derivatives* of a function $\varphi : \mathbb{X} \to \mathbb{R}$ at $x \in \mathbb{X}$ in the direction $v \in \mathbb{X}$ are defined, respectively, by

$$D_+ \varphi(x, v) := \limsup_{t\downarrow 0} \frac{\varphi(x+tv) - \varphi(x)}{t}\,,$$

$$D_- \varphi(x, v) := \liminf_{t\downarrow 0} \frac{\varphi(x+tv) - \varphi(x)}{t}\,.$$

THEOREM 4. *Let $\bar{x} \in \mathbb{X}$ be a local optimal solution for (4), and let*

$$I(\bar{x}) := \{i \in I : f_i(\bar{x}) = 0\}$$

the index set of the active constraints. Then

(i) there exist families

$$\{\partial_{t_0}^{D_-} f_0(\bar{x}) : t_0 \in T_0\} := \{\partial(D_- f_0(\bar{x}, \cdot), t_0) : t_0 \in T_0\}\,,$$

$$\{\partial_{t_i}^{D_+} f_i(\bar{x}) : t_i \in T_i\} := \{\partial(D_+ f_i(\bar{x}, \cdot), t_i) : t_i \in T_i\}\,, \quad \textit{for all } i \in I(\bar{x})$$

of nonempty compact convex sets of $\mathbb{X}^*$ such that

$$D_- f_0(\bar{x}, v) = \min_{t_0 \in T_0} \sigma(v, \partial_{t_0}^{D_-} f_0(\bar{x})) ,$$

$$D_+ f_i(\bar{x}, v) = \min_{t_i \in T_i} \sigma(v, \partial_{t_i}^{D_+} f_i(\bar{x})) , \quad \textit{for all } i \in I(\bar{x}) .$$

(ii) for each $t \in \Pi_{i\in\{0\}\cup I(\bar{x})} T_i$*, there exist* $\lambda_i \geqslant 0$*,* $i \in \{0\} \cup I(\bar{x})$*, not all zero, such that*

$$0 \in \lambda_0 \partial_{t_0}^{D_-} f_0(\bar{x}) + \sum_{i\in I(\bar{x})} \lambda_i \partial_{t_i}^{D_+} f_i(\bar{x}) .$$

Proof. Since f_i, $i \in \{0\} \cup I(\bar{x})$, are Lipschitz functions then also their Dini derivatives are Lipschitz positively homogeneous functions and (i) follows from Theorem 2. Moreover, it is easy to verify that, if $\bar{x} \in \mathbb{X}$ is a local optimal solution for (4), then the system

$$\begin{cases} D_- f_0(\bar{x}, v) < 0 \\ D_+ f_i(\bar{x}, v) < 0 , \quad i \in I(\bar{x}) \end{cases} \tag{5}$$

is not consistent. For each $t \in \Pi_{i\in\{0\}\cup I(\bar{x})} T_i$,

$$\text{barr}\, \partial_{t_0}^{D_-} f_0(\bar{x}) = \text{barr}\, \partial_{t_i}^{D_+} f_i(\bar{x}) = \mathbb{X}$$

then, from Theorem 3, the impossibility of the system (5) is equivalent to state that there exist $\lambda_i \geqslant 0$, $i \in \{0\} \cup I(\bar{x})$, not all zero, such that

$$0 \in \text{cl}\left(\lambda_0 \partial_{t_0}^{D_-} f_0(\bar{x}) + \sum_{i\in I(\bar{x})} \lambda_i \partial_{t_i}^{D_+} f_i(\bar{x}) \right) .$$

Since $\partial_{t_0}^{D_-} f_0(\bar{x})$ and $\partial_{t_i}^{D_+} f_i(\bar{x})$ are compact sets we can omit closure proving the thesis. □

3. Conclusion

In this paper we have presented a dual characterization for a proper positively homogeneous function p such that $p(0) = 0$. Moreover, we have shown that, if p is furthermore a Lipschitz function, such a characterization may be obtained with compact convex sets. Lastly, by means of a generalized Gordan's theorem, we have developed a necessary optimality condition in dual form for a locally Lipschitz extremum problem by using Dini derivatives.

A similar approach has been developed in [2]. A family of compact sets E^* is called an *upper exhauster* of the positively homogeneous function p if

$$p(x) = \inf_{C^* \in E^*} \max_{x^* \in C^*} \langle x^*, x \rangle ; \tag{6}$$

so that the definition of MSL function has been obtained replacing inf with min and max with sup in (6). Nevertheless, the family of functions which admit a representation by means of upper exhauster is less wide than the class of MSL function. It was shown in [2] that any real-valued upper semicontinuous positively homogeneous function, defined on a finite dimensional Euclidean space and vanishing at the origin, can be expressed as an infimum of sublinear continuous real-valued functions. In [9] this result was extended in uniformly convex Banach spaces.

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